

LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS RELATED THE q -GAMMA AND THE q -DIGAMMA FUNCTIONS WITH APPLICATIONS

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ABSTRACT. In this paper we present several new classes of logarithmically completely monotonic functions. Our functions have in common that they are defined in terms of the q -gamma and q -digamma functions. As an applications of this results, some inequalities for the q -gamma and the q -digamma functions are established. Some of the given results generalized theorems due to Alzer and Berg and C.-P. Chen and F. Qi.

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MSC (2010):

1. Introduction

It is well-known that the classical Euler gamma function may be defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt,$$

$\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are called the polygamma functions. It is common knowledge that special functions $\Gamma(x)$, $\psi(x)$ and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are fundamental and important and have much extensive applications in mathematical sciences.

The q -analogue of Γ is defined [[5], pp. 493-496] for $x > 0$ by

$$(1) \quad \Gamma_q(x) = (1-q)^{1-x} \prod_{j=0}^{\infty} \frac{1-q^{j+1}}{1-q^{j+x}}, \quad 0 < q < 1,$$

and

$$(2) \quad \Gamma_q(x) = (q-1)^{1-x} q^{\frac{x(x-1)}{2}} \prod_{j=0}^{\infty} \frac{1-q^{-(j+1)}}{1-q^{-(j+x)}}, \quad q > 1.$$

The q -gamma function $\Gamma_q(z)$ has the following basic properties:

$$(3) \quad \lim_{q \rightarrow 1^-} \Gamma_q(z) = \lim_{q \rightarrow 1^+} \Gamma_q(z) = \Gamma(z),$$

and

$$(4) \quad \Gamma_q(z) = q^{\frac{(x-1)(x-2)}{2}} \Gamma_{\frac{1}{q}}(z).$$

The q -digamma function ψ_q , the q -analogue of the psi or digamma function ψ is defined by

$$\begin{aligned}
 \psi_q(x) &= \frac{\Gamma'_q(x)}{\Gamma_q(x)} \\
 &= -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+x}}{1-q^{k+x}} \\
 &= -\ln(1-q) + \ln q \sum_{k=1}^{\infty} \frac{q^{kx}}{1-q^k} \\
 &= -\ln(1-q) - \int_0^{\infty} \frac{e^{-xt}}{1-e^{-t}} d\gamma_q(t),
 \end{aligned}
 \tag{5}$$

for $0 < q < 1$, where $d\gamma_q(t)$ is a discrete measure with positive masses $-\ln q$ at the positive points $-k \ln q$ for $k \in \mathbb{N}$, more accurately, (see [10])

$$\gamma_q(t) = \sum_{k=1}^{\infty} \delta(t + k \ln q), \quad 0 < q < 1.
 \tag{6}$$

For $q > 1$ and $x > 0$, the q -digamma function ψ_q is defined by

$$\begin{aligned}
 \psi_q(x) &= -\ln(q-1) + \ln q \left[x - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-(k+x)}}{1-q^{-(k+x)}} \right] \\
 &= -\ln(q-1) + \ln q \left[x - \frac{1}{2} - \sum_{k=1}^{\infty} \frac{q^{-kx}}{1-q^{-kx}} \right]
 \end{aligned}$$

Krattenthaler and Srivastava [11] proved that $\psi_q(x)$ tends to $\psi(x)$ on letting $q \rightarrow 1$ where $\psi(x)$ is the ordinary Psi (digamma) function. Before we present the main results of this paper we recall some definitions, which will be used in the sequel.

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0,
 \tag{7}$$

for all $x \in I$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where \mathbb{N} the set of all positive integers.

If the inequality (7) is strict, then f is said to be strictly completely monotonic function.

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n \left(\ln f(x) \right)^{(n)}(x) \geq 0,$$

for all $x \in I$ and $n \in \mathbb{N}$.

A positive function f is said to be logarithmically convex on an interval I , or simply log-convex, if its natural logarithm $\ln f$ is convex, that is, for all $x, y \in I$ and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1-\lambda)y) \leq [f(x)]^\lambda [f(y)]^{1-\lambda}.$$

We note that every logarithmically completely monotonic function is log-convex. From the definition of $\psi_q(x)$, direct differentiation, and the induction we get

$$(8) \quad (-1)^n \psi_q^{(n+1)}(x) = (\ln q)^{n+2} \sum_{k=1}^n \frac{k^{n+1} q^{kx}}{1 - q^k} > 0,$$

which implies that the function ψ_q' is strictly completely monotonic function on $(0, \infty)$, for $q \in (0, 1)$. The relation (4) and the definition of the q -digamma function (5) give

$$(9) \quad \psi_q(x) = \psi_{\frac{1}{q}}(x) + \frac{2x-3}{2} \ln q,$$

for $q > 1$. Thus, implies that the function $\psi_q'(x)$ is strictly completely monotonic in $(0, \infty)$ for $q > 1$, and consequently the function $\psi_q(x)$ is strictly completely monotonic on $(0, \infty)$.

It is the aim of this paper to provide several new classes of logarithmically completely monotonic functions. The functions we study have in common that they are defined in terms of q -gamma and q -digamma functions. In the next section we collect some lemmas. Our monotonicity theorems are stated and proved in sections 3.

2. Useful lemmas

We begin this section with the following useful lemmas which are needed to completes the proof of the main theorems.

The following monotonicity theorem is proved in [2].

Lemma 1. *Let n be a natural number and c be a real number. The function $x^c |\psi^n(x)|$ is decreasing on $(0, \infty)$ if and only if $c \leq n$.*

Lemma 2. [14] *Let $f, g : [a, b] \rightarrow \mathbb{R}$ two continuous functions which are differentiable on (a, b) . Further, let $g' \neq 0$ on (a, b) . If $\frac{f'}{g'}$ is increasing (or decreasing) on (a, b) , then the functions $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$ are also increasing (or decreasing) on (a, b) .*

The next lemma is given in [1, 16].

Lemma 3. *The function ψ_q , $q > 0$ has a uniquely determined positive zero, which we denoted by $x_0 = x_0(q) \in (1, 2)$.*

A proof for the following lemma can be found in [8].

Lemma 4. *Let f be a positive function. If f' is completely monotonic function, then $\frac{1}{f}$ is logarithmically completely monotonic function.*

The following inequality for completely monotonic functions is due to Kimberling [12].

Lemma 5. *If a function f , defined on $(0, \infty)$, is continuous and completely monotonic and maps $(0, \infty)$ into $(0, 1)$, then $\ln f$ is super-additive, that is for all $x, y > 0$ we have*

$$\ln f(x) + \ln f(y) \leq \ln f(x+y) \text{ or } f(x)f(y) \leq f(x+y).$$

3. The main results

In 1997, Merkle [15] proved that the function $\frac{(\Gamma(x))^2}{\Gamma(2x)}$ is log-convex on $(0, \infty)$. Recently, Alzer and Berg [4], presented a substantial generalization. They established that the function

$$(10) \quad \frac{(\Gamma(ax))^\alpha}{(\Gamma(bx))^\beta}, \quad 0 < b < a, \quad (\text{or } 0 < a < b)$$

is completely monotonic on $(0, \infty)$, if and only if, $\alpha \leq 0$, (or $\alpha \geq 0$) and $\alpha a = \beta b$. The main objective of the next Theorem extend and generalize this results. An application of their result leads to sharp upper and lower bounds for $\frac{\Gamma_q^2(x)}{\Gamma_q(2x)}$ in terms of the ψ_q -function.

Theorem 1. *Let $0 < q < 1$ and $0 < a < b$. Then the function $\frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta}$ is logarithmically completely monotonic on $(0, \infty)$ if and only if $\alpha \geq 0$ and $\alpha a = \beta b$.*

Proof. Let $q \in (0, 1)$. Assume that the function $\frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta}$ is logarithmically completely monotonic on $(0, \infty)$. By definition, this gives us that for all real $x > 0$,

$$(11) \quad \left[\ln \frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta} \right]' = \alpha a \psi_q(ax) - \beta b \psi_q(bx) \leq 0.$$

Now we prove $\alpha \geq 0$. By contradiction, suppose that $\alpha < 0$ such that $\alpha a = \beta b$. Since the function $\psi_q(x)$ is strictly increasing on $(0, \infty)$ we have

$$(12) \quad \alpha a (\psi_q(ax) - \psi_q(bx)) > 0,$$

for $x > 0$ and $0 < a < b$. On the other hand, by inequality (11) we obtain that

$$(13) \quad \alpha a (\psi_q(ax) - \psi_q(bx)) \leq 0.$$

Inequalities (12) and (13) implies that the constant $\alpha \geq 0$.

On the other hand, using the fact

$$\lim_{x \rightarrow +\infty} \psi_q(x) = -\ln(1 - q),$$

for $q \in (0, 1)$, and equality (11) we obtain

$$(\alpha a - \beta b)(-\ln(1 - q)) \leq 0.$$

and consequently

$$(\alpha a - \beta b) \leq 0.$$

On the other hand, using the definition we get

$$\left[\ln \frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta} \right]^{(2)} = \alpha a^2 \psi_q^{(1)}(ax) - \beta b^2 \psi_q^{(1)}(bx) \geq 0.$$

If $x \rightarrow 0^+$, we obtain

$$(14) \quad \alpha a^2 - \beta b^2 \geq 0,$$

since $0 < \frac{1}{b} < \frac{1}{a}$ we have $\alpha a \geq \beta b$, this inequality and inequality (14) leads to $\alpha a = \beta b$.

Now we show that the function $\frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta}$ is logarithmically completely monotonic on $(0, \infty)$ for $\alpha, \beta \geq 0$, $0 < a < b$ such that $\alpha a = \beta b$. Let $n = 1$, since the function $\psi_q(x)$ is strictly increasing on $(0, \infty)$ we have

$$(15) \quad (-1) \left(\ln \frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta} \right)' = \alpha a (\psi_q(bx) - \psi_q(ax)) \geq 0.$$

For $n \geq 1$, we get

$$(16) \quad (-1)^{n+1} \left(\ln \frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta} \right)^{(n+1)} = (-1)^n \alpha a \left(b^n \psi_q^{(n)}(bx) - a^n \psi_q^{(n)}(ax) \right).$$

Since the function $\psi_q'(x)$ is strictly completely monotonic on $(0, \infty)$, we have for $n \in \mathbb{N}$ and $q > 0$ we have

$$(17) \quad |\psi_q^{(n)}(x)| = (-1)^{n+1} \psi_q^{(n)}(x).$$

Thus

$$\begin{aligned} (-1)^{n+1} x^n \left(\ln \frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta} \right)^{(n+1)} &= (-1)^n \alpha a x^n \left(b^n \psi_q^{(n)}(bx) - a^n \psi_q^{(n)}(ax) \right) \\ &= (-1)^n \alpha a \left((xb)^n (-1)^n |\psi_q^{(n)}(bx)| - (ax)^n (-1)^n |\psi_q^{(n)}(ax)| \right) \\ &= \alpha a \left((ax)^n |\psi_q^{(n)}(ax)| - (bx)^n |\psi_q^{(n)}(bx)| \right), \end{aligned}$$

and the last expression is nonnegative by Lemma 1. Hence, for $q \in (0, 1)$ and $n \in \mathbb{N}$

$$(-1)^n \left(\ln \frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta} \right)^{(n)} \geq 0.$$

The proof of Theorem 1 is complete. ■

Corollary 1. *Let $q > 1$ and $0 < a < b$. If, $\alpha \geq 0$ and $\alpha a = \beta b$, Then the function $\frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta}$ is logarithmically completely monotonic on $(0, \infty)$.*

Proof. Follows immediately by Theorem 1 and equality (9). ■

Corollary 2. *Let $q > 0$ and $0 < a < b$, the following inequalities*

$$(18) \quad \exp [\alpha a (x - x_1) (\psi_q(ax_1) - \psi_q(bx_1))] \leq \frac{(\Gamma_q(bx_1))^\beta}{(\Gamma_q(ax_1))^\alpha} \frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta} \leq 1,$$

holds for all $\alpha, \beta \geq 0$ such that $\alpha a = \beta b$. and $x > x_1 > 0$ In particular, the following inequalities holds true for every integer $n \geq 1$:

$$(19) \quad \exp \left[2q(n-1) \frac{\ln q}{1-q} \right] \leq \frac{\Gamma_q^2(n)}{\Gamma_q(2n)} \leq 1.$$

Proof. Let $q > 0$ and $0 < a < b$. We suppose that $\alpha, \beta \geq 0$ such that $\alpha a = \beta b$ and define the function $h_{\alpha,\beta}(q; x)$ by

$$h_{\alpha,\beta}(q; x) = \frac{(\Gamma_q(bx_1))^\beta (\Gamma_q(ax))^\alpha}{(\Gamma_q(ax_1))^\alpha (\Gamma_q(bx))^\beta}$$

where $0 < x_1 < x$, and $H_{\alpha,\beta,q}(x) = \ln h_{\alpha,\beta}(q; x)$. Since the function $h_{\alpha,\beta}(q; x)$ is logarithmically completely monoyonic on $(0, \infty)$ for $\alpha \geq 0$ and $\alpha a = \beta b$, we conclude that the logarithmic derivative $\frac{(h_{\alpha,\beta}(q; x))'}{h_{\alpha,\beta}(q; x)}$ is increasing on $(0, \infty)$. By Lemma 2 we deduce that the function $\frac{H_{\alpha,\beta,q}(x)}{x - x_1}$ is increasing for all $0 < x_1 < x$. By l'Hospital's rule and (11) it is easy to deduce that

$$\lim_{x \rightarrow x_1} \frac{H_{\alpha,\beta,q}(x)}{x - x_1} = \alpha a (\psi_q(ax_1) - \psi_q(bx_1)),$$

from which follows the right side inequality of (18).

As $h_{\alpha,\beta}(q; x)$ is logarithmically completely monotonic on $(0, \infty)$, we deduce that $h_{\alpha,\beta}(q; x)$ is decreasing on $(0, \infty)$. The following inequality hold true for all $0 < x_1 < x$:

$$h_{\alpha,\beta}(q; x) \leq h_{\alpha,\beta}(q; x_1) = 1,$$

we conclude the left side inequality of (18).

Taking $\alpha = b = 2$, $\beta = a = 1$ and $x_1 = 1$ in (18) and using the recurrence formula of ψ_q [[9], p. 1245, Theorem 4.4]

$$(20) \quad \psi_q^{(n-1)}(x+1) - \psi_q^{(n-1)}(x) = -\frac{d^{n-1}}{dx^{n-1}} \left(\frac{q^x}{1 - q^x} \right) \ln q$$

we obtain the inequalities (19). ■

The main purpose of the next Theorem is to present monotonicity properties of the function

$$(21) \quad g_\beta(q; x) = \frac{1}{1+q} \left[\frac{\Gamma_{q^2}(x + \frac{1}{2})}{\Gamma_{q^2}(x+1)} \right]^2 \exp \left[\frac{\beta(1-q^2)q^{2x}}{2(1-q^{2x})} + \psi_q(2x) \right],$$

where $q \in (0, 1)$ and $x > 0$.

It is worth mentioning that Ai-Jun Li and Chao-Ping Chen [13] considered the function

$$(22) \quad g_\beta(x) = \frac{1}{2} \left[\frac{\Gamma(x + \frac{1}{2})}{\Gamma(x+1)} \right]^2 \exp \left[\frac{\beta}{2x} + \psi(2x) \right]$$

which is a special case of the function $g_\beta(q; x)$ on letting $q \rightarrow 1$ and proved that $g_\beta(x)$ is logarithmically completely monotonic on $(0, \infty)$ if $\beta \geq \frac{13}{12}$. The objective of this Theorem is to generalize this result.

Theorem 2. *Let $q \in (0, 1)$. The function $g_\beta(q; x)$ defined by (21) is logarithmically completely monotonic on $(0, \infty)$ if $\beta \geq \frac{-13 \ln q}{6(1-q^2)}$.*

Proof. It is clear that

$$\ln g_\beta(q; x) = 2 \ln \Gamma_{q^2}(x + \frac{1}{2}) - 2 \ln \Gamma_{q^2}(x+1) + \psi_q(2x) + \frac{\beta(1-q^2)q^{2x}}{2(1-q^{2x})} - \ln(q+1).$$

Using the q -analogue of Legendre's duplication formula (??) we get

$$\ln g_\beta(q; x) = 2 \ln \Gamma_{q^2}(x + \frac{1}{2}) - 2 \ln \Gamma_{q^2}(x + 1) + \frac{1}{2} \psi_{q^2}(x) + \frac{1}{2} \psi_{q^2}(x + \frac{1}{2}) + \frac{\beta(1 - q^2)q^{2x}}{2(1 - q^{2x})}.$$

In view of (20) and (5) we obtain that

$$\begin{aligned} (-1)^n (\ln g_\beta(q; x))^{(n)} &= (-1)^n \left[2\psi_{q^2}^{(n-1)}(x + \frac{1}{2}) - 2\psi_{q^2}^{(n-1)}(x + 1) + \frac{1}{2}\psi_{q^2}^{(n)}(x) + \frac{1}{2}\psi_{q^2}^{(n)}(x + \frac{1}{2}) \right. \\ &\quad \left. - \frac{\beta(1 - q^2)}{4 \ln q} (\psi_{q^2}^{(n)}(x + 1) - \psi_{q^2}^{(n)}(x)) \right] \\ &= \frac{1}{2} \int_0^\infty \frac{e^{-xt}}{e^t - 1} \Phi_{\beta, q}(t) d\gamma_{q^2}(t) \end{aligned}$$

where

$$\begin{aligned} \Phi_{\beta, q}(t) &= \left(\frac{-\beta(1 - q^2)}{2 \ln q} - 1 \right) t e^t + (4 - t) e^{\frac{t}{2}} + \frac{\beta(1 - q^2)}{2 \ln q} t - 4 \\ &= \sum_{n=0}^\infty \frac{t^{n+1}}{n!} \left[\frac{-\beta(1 - q^2)}{2 \ln q} - 1 - \frac{1}{2^n} \right] + 4 \sum_{n=1}^\infty \frac{t^n}{2^n n!} + \frac{\beta(1 - q^2)}{2 \ln q} t \\ &= \sum_{n=1}^\infty \frac{t^{n+1}}{n!} \left[\frac{-\beta(1 - q^2)}{2 \ln q} - 1 - \frac{1}{2^n} \right] + 4 \sum_{n=0}^\infty \frac{t^{n+1}}{2^{n+1} (n+1)!} - 2t \\ &= \sum_{n=1}^\infty \frac{t^{n+1}}{n!} \left[\frac{-\beta(1 - q^2)}{2 \ln q} - 1 - \frac{1}{2^n} + \frac{1}{(n+1)2^{n-1}} \right] \end{aligned}$$

Since the $\max \left(\frac{1}{2^n} + \frac{1}{(n+1)2^{n-1}} \right) = \frac{1}{12}$, we conclude that $\beta \geq \frac{-13 \ln q}{6(1 - q^2)}$. The proof is completed. \blacksquare

Theorem 3. Let $q > 0$, the function $\frac{1}{\psi_q(x)}$ is Logarithmically completely monotonic on (x_0, ∞) .

Proof. Since the function ψ_q' is completely monotonic function on $(0, \infty)$, and the function ψ_q is increasing on $(0, \infty)$ and a uniquely determined zero on $(0, \infty)$ by Lemma 3. We conclude that the function $\psi_q(x) > 0$ for all $x > x_0$, and consequently the function $\frac{1}{\psi_q(x)}$ is Logarithmically completely monotonic on (x_0, ∞) , by Lemma 4. \blacksquare

Corollary 3. Let $q > 0$ and $a > 1$. The following inequality

$$(23) \quad [\psi_q(x)]^{\frac{1}{a}} [\psi_q(y)]^{1 - \frac{1}{a}} \leq \psi_q \left[\frac{x}{a} + \left(1 - \frac{1}{a} \right) y \right]$$

holds for all $x > x_0$ and $y > x_0$. In particular, the following inequality holds

$$(24) \quad [\psi_q(2)]^{a-1} \leq \frac{[\psi_q(u+1)]^a}{\psi_q(a(x-1)+2)}$$

for all $a > 1$ and $u > \frac{-2}{a} + 1$.

Proof. Let $q \in (0, 1)$, $a > 1$, $x > x_0$ and $y > x_0$. By theorem 3 we obtain that the function $\frac{1}{\psi_q(x)}$ is log-convex on (x_0, ∞) . Thus,

$$[\psi_q(x)]^{\frac{1}{p}} [\psi_q(y)]^{\frac{1}{q}} \leq \psi_q \left[\frac{x}{p} + \frac{y}{q} \right],$$

where $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. If $p = a$ and $q = \frac{a}{a-1}$. Then we get the inequality (23). Now, let $y = 2$ and $x = a(u - 1) + 2$ we obtain the inequality (24). ■

Remark 1. Replacing u by $n \in \mathbb{N}$ and a by 2 in inequality (24) and using the identity

$$\psi_q(n+1) = \frac{\ln q}{1-q} \gamma_q - \ln q H_{n,q}$$

where $\gamma_q = \frac{1-q}{\ln q} \psi_q(1)$ is the q -analogue of the Euler-Mascheroni constant [17] and $H_{n,q}$ is the q -analogue of Harmonic number is defined by [18] as

$$H_{n,q} = \sum_{k=1}^n \frac{q^k}{1-q^k}, \quad n \in \mathbb{N},$$

we obtain

$$(25) \quad \psi_q^2(2) \psi_q(2n) \leq \left[\frac{\ln q}{1-q} \gamma_q - \ln q H_{n,q} \right]^2, \quad n \in \mathbb{N}.$$

Theorem 4. Let $q > 0$ The function $\Gamma_q(x)$ is logarithmically completely monotonic on $(0, x_0)$. So, the following inequality

$$(26) \quad \Gamma_q(x+1) \Gamma_q(y+1) \leq \Gamma_q(x+y+2)$$

holds for all $x, y \in (0, 1)$.

Proof. Proving by induction that

$$(-1)^n (\ln \Gamma_q(x))^{(n)} \geq 0, \quad \text{for all } n \in \mathbb{N}.$$

For $n = 1$, we get

$$(-1)(\ln \Gamma_q(x))^{(1)} = \psi_q(x) \geq 0, \quad \text{for all } x \in (0, x_0).$$

Suppose that

$$(-1)^k (\ln \Gamma_q(x))^{(k)} \geq 0, \quad \text{for all } 1 \leq k \leq n \text{ and } x \in (0, x_0).$$

Since the function ψ_q' is completely monotonic on $(0, \infty)$, for $q > 0$ we get

$$(-1)^{n+1} (\ln \Gamma_q(x))^{(n+1)} = (-1)^n \psi_q^{(n+1)}(x) \geq 0.$$

We note that every logarithmically completely monotonic function is completely monotonic [7], and the function $\Gamma_q(x) \leq 1$ if and only if $1 \leq x \leq 2$. The above results imply that the function $\Gamma_q(x)$ is completely monotonic on $(0, x_0)$ and maps $(1, 2)$ into $(0, 1)$, applying Lemma 5, we conclude the asserted inequality (26). ■

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